# A CALCULATION OF THE MULTIPLICATIVE CHARACTER

J. KAAD

Department of Mathematical Sciences, University of Copenhagen
Universitetsparken 5, DK-2100 Copenhagen, Denmark

**Abstract**

We give a formula, in terms of products of commutators, for the application of the odd multiplicative character to higher Loday symbols. On our way we construct a product on the relative $K$-groups and investigate the multiplicative properties of the relative Chern character.

email: `kaad@math.ku.dk`.

## Contents

## 1. Introduction

To each finitely summable Fredholm module $(F, H)$ over a $\mathbb{C}$-algebra $A$, A. Connes and M. Karoubi associate a multiplicative character on algebraic $K$-theory

$$M_F : K_n(A) \to \mathbb{C}/(2\pi i)^{\lceil \frac{n}{2} \rceil}\mathbb{Z}$$

The construction uses the relative $K$-groups of a unital Banach algebra and the relative Chern character with values in continuous cyclic homology. It can be understood as a pairing between the abelian group generated by finitely summable Fredholm modules and algebraic $K$-theory, [12].

In the case where $n = 1$ the multiplicative character has a direct interpretation as a Fredholm determinant, see [12]. Likewise, in the case where $n = 2$ the multiplicative character coincides with the determinant invariant as defined in [4, 5]. See also [23]. In particular, when the $\mathbb{C}$-algebra $A$ is commutative we have the explicit formula

$$M_F([e^a] * [e^b]) = -\mathrm{Tr}[PaP, PbP] \in \mathbb{C}/(2\pi i)\mathbb{Z} \tag{1}$$

for the application of the multiplicative character to the Loday product $[e^a] * [e^b] \in K_2(A)$. Furthermore this implies the independence of the character under trace class perturbations, [15]. Note that a description of the multiplicative character in terms of a (different) central extension has also been obtained in [12]. We could thus try to think of the multiplicative character as an extension of the determinant invariant to higher $K$-theory. The aim of the present paper is then to find an analogue of the formula (1) in higher dimensions. This pursuit could be justified by the large amount of research which focus on the quantity $\mathrm{Tr}[PaP, PbP] \in \mathbb{C}$, see [1, 2, 8, 9, 14], among others. We would also like to mention the use of the determinant invariant in relation with the Szegö limit theorem, [6, 7].

Let us fix an odd $2p$-summable Fredholm module $(F, H)$ over a *commutative* Banach algebra. The interior Loday product makes the direct sum of algebraic $K$-groups $\oplus_{n=1}^{\infty} K_n(A)$ into a graded commutative ring, see [18]. We can thus consider the application of the multiplicative character to the Loday product $[e^{a_0}] * \ldots * [e^{a_{2p-1}}] \in K_{2p}(A)$. Here $a_0, \ldots, a_{2p-1} \in M_\infty(A)$. The main result of the present paper is then the concrete formula

$$\begin{aligned} &M_F([e^{a_0}] * \ldots * [e^{a_{2p-1}}]) \\ &\quad = (-1)^p c_p \sum_{s \in SE_{2p-1}} \mathrm{sgn}(s) \mathrm{Tr}([P\mathrm{TR}(a_0)P, P\mathrm{TR}(a_{s(1)})P] \cdot \ldots \cdot \\ &\qquad\qquad [P\mathrm{TR}(a_{s(2p-2)})P, P\mathrm{TR}(a_{s(2p-1)})P]) \in \mathbb{C}/(2\pi i)^p\mathbb{Z} \end{aligned} \tag{2}$$

Here $c_p \in \mathbb{Q}$ is a constant and $SE_{2p-1} \subseteq \Sigma_{2p-1}$ is the subset of permutations satisfying $s(2i) < s(2i+1)$. The operator $P = (F+1)/2$ is the projection associated with the Fredholm module $(F, H)$. This shows that the multiplicative character is calculizable on the subgroup of $K_{2p}(A)$ generated by Loday products of elements in the connected component of the identity. Note that the commutativity assumption serves to ensure the existence of the *interior* Loday product which is needed for the calculation to make sense.

Now, for each $2p$-tuple $(a_0,\ldots,a_{2p-1}) \in M_\infty(A)$ we define the complex number

$$\langle a_0,\ldots,a_{2p-1}\rangle = (-1)^p c_p \sum_{s\in SE_{2p-1}} \mathrm{sgn}(s)\mathrm{Tr}([P\mathrm{TR}(a_0)P, P\mathrm{TR}(a_{s(1)})P]\cdot\ldots\cdot [P\mathrm{TR}(a_{s(2p-2)})P, P\mathrm{TR}(a_{s(2p-1)})P]) \in \mathbb{C}$$

Let us then reflect a bit on what we have obtained. First of all, choosing a different "logarithm" for $e^{a_i} \in GL_0(A)$, that is some $b_i \in M_\infty(A)$ with $e^{b_i} = e^{a_i}$, we get that the difference

$$\langle a_0,\ldots,a_i,\ldots,a_{2p-1}\rangle - \langle a_0,\ldots,b_i,\ldots,a_{2p-1}\rangle \in (2\pi i)^p\mathbb{Z} \tag{3}$$

is in the additive group $(2\pi i)^p\mathbb{Z}$. That is, the quantity (3) is essentially the index of some Fredholm operator. This is an immediate Corollary of the formula (2). Furthermore we get a couple of desirable properties straight from the corresponding properties of the Loday product and the algebraic $K$-groups, [18]. For example, the map

$$GL(A)^{2p} \to \mathbb{C}/(2\pi i)^p\mathbb{Z} \qquad (g_0,\ldots,g_{2p-1}) \mapsto M_F([g_0]*\ldots*[g_{2p-1}])$$

is multilinear and it sends each tuple with an elementary entrance, $g_i \in E(A)$, to zero.

The similiarity of the trace formula in (2) with the expression in the bivariant case, makes us expect the following generalizations : First of all the quantity $\langle a_0,\ldots,a_{2p-1}\rangle \in \mathbb{C}$ should be invariant under perturbations of the operators $Pa_iP$ by elements in the Schatten ideal $\mathcal{L}^p(H)$. Furthermore, under suitable conditions, we expect our form to be expressible by means of an integral over the joint essential spectrum of the operators in question.

Finally, we would like to explain briefly how the main result is obtained. In order to calculate the multiplicative character of some element $[x] \in K_{2p}(A)$ the first obstacle is to construct a lift in relative $K$-theory,

$$[\gamma] \in K_{2p}^{\mathrm{rel}}(A) \qquad \theta[\gamma] = [x]$$

In the special case where the element $[x] \in K_{2p}(A)$ is given by a product of contractible invertible operators this is accomplished by the construction of an explicit product on the relative $K$-groups. The product makes the direct sum of relative $K$-groups $\oplus_{n=1}^\infty K_n^{\mathrm{rel}}(A)$ into a graded commutative ring and the map $\theta : \oplus_{n=1}^\infty K_n^{\mathrm{rel}}(A) \to \oplus_{n=1}^\infty K_n(A)$ becomes a homomorphism of graded rings (recall that $A$ is assumed to be commutative). The question of finding the lift $[\gamma] \in K_{2p}^{\mathrm{rel}}(A)$ then reduces to lifting each of the elements $[g_i] \in K_1(A)$. This is possible by the contractibility assumption. The construction of the product is carried out in Section 3. On our way we also express the second relative $K$-group as the second homology group of a certain simplicial set.

Having found the lift $[\gamma] \in K_{2p}^{\mathrm{rel}}(A)$ the next problem is to calculate the relative Chern character of the lift

$$\mathrm{ch}^{\mathrm{rel}} : K_{2p}^{\mathrm{rel}}(A) \to HC_{2p-1}(A) \qquad \mathrm{ch}^{\mathrm{rel}}[\gamma] = ?$$

Following the same vein of ideas we show in Section 4 that the relative Chern character is a homomorphism of graded rings. This should be understood in the following sense : The relative Chern character has degree minus one, so the corresponding product in continuous cyclic homology has degree plus one,

$$* : HC_{n-1}(A) \otimes_{\mathbb{C}} HC_{m-1}(A) \to HC_{n+m-1}(A) \qquad x * y = x \times (sN)(y)$$

See also [19]. The calculation in question thus reduces to the case of $\mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(A) \to HC_0(A)$. The elements in $K_1^{\mathrm{rel}}(A)$ are represented by smooth maps $\sigma : [0,1] \to GL(A)$ mapping 0 to the identity $1 \in GL(A)$ and the relative Chern character essentially determines the corresponding logarithm of the endpoint $\sigma(1) \in GL_0(A)$.

The desired formula (2) can now be obtained from combinatorial considerations on the index cocycle associated with Fredholm modules over commutative algebras. This is carried out in Section 5 where the main Theorem is presented.

We begin by giving an account of the various product structures which will be used throughout the paper.

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## 2. A preliminary on various product structures in homology

2.1. **The exterior shuffle product.** Let $A$ and $B$ be unital Banach algebras. We let $A\hat{\otimes}B$ denote the projective tensor product of $A$ and $B$ in the sense of Grothendieck, [13]. The definition of the simplicial sets $R_p(A)$ can be found in Section 3.

For each $p, q \in \mathbb{N}$ we fix an isomorphism $\varphi : A^p \otimes_{\mathbb{Z}} B^q \to (A \otimes_{\mathbb{Z}} B)^{pq}$ of $(A \otimes_{\mathbb{Z}} B)$-bimodules. We then have the associated group homomorphisms

$$\otimes_\varphi : GL_p(A) \times GL_q(B) \to GL_{pq}(A \otimes_{\mathbb{Z}} B) \qquad \text{and}$$
$$\hat{\otimes}_\varphi = \iota \circ \otimes_\varphi : GL_p(A) \times GL_q(B) \to GL_{pq}(A\hat{\otimes}B)$$

Here $\iota : GL_{pq}(A\otimes_{\mathbb{Z}}B) \to GL_{pq}(A\hat{\otimes}B)$ is induced by the "identity" homomorphism $\iota : A\otimes_{\mathbb{Z}}B \to A\hat{\otimes}B$.

A pointwise version of the completed tensor product yields a map of simplicial sets

$$\hat{\otimes}_\varphi : R_p(A) \times R_q(B) \to R_{pq}(A\hat{\otimes}B) \qquad (\sigma, \tau) \mapsto (t \mapsto \sigma(t)\hat{\otimes}_\varphi\tau(t))$$

Composition with the shuffle map [21]

$$\mathrm{sh} : C_*(R_p(A)) \otimes C_*(R_q(B)) \to C_*(R_p(A) \times R_q(B))$$

therefore equips us with a chain map

$$\times_\varphi = \hat{\otimes}_\varphi \circ \mathrm{sh} : C_*(R_p(A)) \otimes C_*(R_q(B)) \to C_*(R_{pq}(A\hat{\otimes}B))$$

We will refer to the sum of smooth maps

$$\sigma \times_\varphi \tau = \sum_{(\mu,\nu)\in\Sigma_{n,m}} \mathrm{sgn}(\mu,\nu) s_{\nu(m-1)} \dots s_{\nu(0)}(\sigma)\hat{\otimes}_\varphi s_{\mu(n-1)} \dots s_{\mu(0)}(\tau)$$

as the *exterior shuffle product* of $\sigma \in R_p(A)_n$ and $\tau \in R_q(B)_m$. Here $\Sigma_{(n,m)} \subseteq \Sigma_{n+m}$ denotes the set of $(n,m)$-shuffles. Note that it follows by Lemma 3.9 that the induced map on homology

$$\times : H_n(R_p(A)) \otimes H_m(R_q(B)) \to H_{n+m}(R_{pq}(A\hat{\otimes}B))$$

is independent of the choice of isomorphism $\varphi : A^p \otimes_{\mathbb{Z}} B^q \to (A \otimes_{\mathbb{Z}} B)^{pq}$.

2.2. **The exterior wedge product in Lie algebra homology.** Let $A$ and $B$ be unital Banach algebras. For each $n \in \mathbb{N}$ we let $\Lambda_n A$ denote the kernel of the map

$$S : \underbrace{A\hat{\otimes}\ldots\hat{\otimes}A}_{n} \to \underbrace{A\hat{\otimes}\ldots\hat{\otimes}A}_{n} \qquad S(a_1 \otimes \ldots \otimes a_n) = \sum_{\sigma\in\Sigma_n} \mathrm{sgn}(\sigma) a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)}$$

Remark that the Banach space $\Lambda_n A$ identifies with the quotient of $A^{\hat{\otimes} n}$ by the usual action of the symmetric group.

By the *continuous* Lie algebra homology of the (unital) Banach algebra $A$ we will then understand the homology of the chain complex $(\Lambda_* A, \delta)$. Here $\delta : \Lambda_n A \to \Lambda_{n-1} A$ is the Chevalley-Eilenberg boundary map.

We let

$$\cdot \otimes 1_B : \Lambda_* A \to \Lambda_*(A\hat{\otimes}B) \quad \text{and} \quad 1_A \otimes \cdot : \Lambda_* B \to \Lambda_*(A\hat{\otimes}B)$$

denote the chain maps obtained by functoriality from the continuous algebra homomorphisms

$$x \mapsto x \otimes 1_B \quad \text{and} \quad y \mapsto 1_A \otimes y$$

We then have a chain map

$$\wedge^E : \Lambda_* A \otimes \Lambda_* B \to \Lambda_*(A\hat{\otimes}B)$$

defined by

$$x \otimes y \mapsto (x \otimes 1_B) \wedge (1_A \otimes y) \qquad x \in \Lambda_n A\,,\ y \in \Lambda_m B$$

For each $x \in \Lambda_n A$ and each $y \in \Lambda_m B$ we will refer to the element

$$x \wedge^E y := (x \otimes 1_B) \wedge (1_A \otimes y)$$

as the *exterior wedge product* of $x$ and $y$. We let

$$\wedge^E : H_n^{\mathrm{Lie}}(A) \otimes H_m^{\mathrm{Lie}}(B) \to H_{n+m}^{\mathrm{Lie}}(A\hat{\otimes}B)$$

denote the induced map on continuous Lie algebra homology. The exterior wedge product, thus defined, is seen to be associative and graded commutative on the level of complexes.

2.3. **The exterior product of degree one in cyclic homology.** Let $A$ and $B$ be unital Banach algebras. We let

$$\times : \big(C(A) \otimes C(B)\big)_* \to C_*(A\hat{\otimes}B)$$

denote the exterior shuffle product on the continuous Hochschild complex, [19, Section 4.2]. Furthermore, we let $(C_*^\lambda(A), b)$ denote the *continuous* cyclic complex. Thus in each degree $n \in \mathbb{N}\cup\{0\}$ we have a Banach space $C_n^\lambda(A)$, [12, 17]. Remark that the image $\mathrm{Im}(1-t) \subseteq A\hat{\otimes}A^{\hat{\otimes} n}$ is closed since it coincides with the kernel of the norm operator $N : A\hat{\otimes}A^{\hat{\otimes} n} \to A\hat{\otimes}A^{\hat{\otimes} n}$.

By the exterior product of degree one in continuous cyclic homology we will understand the map

$$* : C_n^\lambda(A) \otimes C_m^\lambda(B) \to C_{n+m+1}^\lambda(A\hat{\otimes}B)$$

defined by

$$x * y = x \times (sNy) \qquad x \in C_n^\lambda(A)\,,\ y \in C_m^\lambda(B)$$

Here $N : C_m(B) \to C_m(B)$ is the norm operator $N = 1+t+\ldots+t^m$ and $s : C_m(B) \to C_{m+1}(B)$ is the extra degeneracy.

We will need to show that the product is well defined. For this, consider the map

$$E : C_n(A) \to \Lambda_{n+1} M_{n+1}(A) \qquad (a_0, \ldots, a_n) \mapsto E_{12}(a_0) \wedge \ldots \wedge E_{(n+1)1}(a_n)$$

where $E_{ij}(a)$ denotes the elementary matrix with $a \in A$ in position $(i,j)$ and zeros elsewhere, [20, 27].

From Theorem 4.4 and Theorem 4.5 we then get the equality

$$x * y = (\mathrm{TR} \circ \varepsilon)(E(x) \wedge^E E(y)) \qquad x \in C_n(A)\,,\, y \in C_m(B)$$

where $\varepsilon : \Lambda_* M_k(A) \to C^\lambda_{*-1}(M_k(A))$ and $\mathrm{TR} : C^\lambda_*(M_k(A)) \to C^\lambda_*(A)$ denote the antisymmetrization map and the generalized trace on continuous cyclic homology respectively. It follows that the product is well defined and that it is associative and graded commutative on the level of complexes.

Lastly, the Hochschild boundary is a (shifted) graded derivation with respect to the product

$$b(x * y) = (bx) * y + (-1)^{\deg(x)+1} x * (by)$$

It follows that our multiplication descends to an exterior product of degree one on continuous cyclic homology

$$* : HC_n(A) \otimes_{\mathbb{C}} HC_m(B) \to HC_{n+m+1}(A \hat{\otimes} B)$$

In the case where the unital Banach algebra $A$ is commutative we get an interior product

$$* : HC_n(A) \otimes_{\mathbb{C}} HC_m(A) \to HC_{n+m+1}(A)$$

by composition of the exterior product with the map induced by the multiplication $\nabla_* : A \hat{\otimes} A \to A$.

For further details on the constructions given in this section we refer to [19, 28].

## 3. An exterior product on the relative $K$-theory of Banach algebras

Let $A$ be a unital Banach algebra. Before giving the construction of the exterior product, we recall the definition of the relative $K$-groups, as introduced by M. Karoubi, [17].

To this end, for each $n \in \mathbb{N}_0$ we let $\Delta^n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \,|\, t_i \geq 0\,,\, \sum_{i=1}^n t_i \leq 1\}$ denote the standard $n$-simplex and we let $\mathbf{0}, \ldots, \mathbf{n} \in \Delta^n$ denote the vertices.

For each $p \in \mathbb{N} \cup \{\infty\}$ we then associate a simplicial set $R_p(A)$. In degree $n \in \mathbb{N}_0$ it is given by the set of normalized continuous maps

$$\sigma : \Delta^n \to GL_p(A) \qquad \sigma(\mathbf{0}) = 1_p$$

The face operators and degeneracy operators are given by

$$d_i(\sigma)(t_1, \ldots, t_{n-1}) = \begin{cases} \sigma(1 - \sum_{j=1}^{n-1} t_j, t_1, \ldots, t_{n-1}) \cdot \sigma(\mathbf{1})^{-1} & \text{for} \quad j = 0 \\ \sigma(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) & \text{for} \quad j \in \{1, \ldots, n\} \end{cases}$$

$$s_j(\sigma)(t_1, \ldots, t_{n+1}) = \begin{cases} \sigma(t_2, \ldots, t_{n+1}) & \text{for} \quad j = 0 \\ \sigma(t_1, \ldots, t_{i-1}, t_i + t_{i+1}, \ldots, t_{n+1}) & \text{for} \quad j \in \{1, \ldots, n\} \end{cases}$$

Remark the extra factor $\sigma(\mathbf{1})^{-1}$ in the expression for $d_0 : R_p(A)_n \to R_p(A)_{n-1}$. We will often refer to the simplicial set $R_\infty(A)$ by $R(A)$.

The fundamental group of the pointed Kan complex $R_p(A)$ is given by

$$\pi_1(R_p(A)) = R_p(A)_1/\sim$$

where $\sim$ denotes the equivalence relation of homotopies with fixed end points. The group structure is given by pointwise multiplication. For each $p \in \{3, 4, \ldots\} \cup \{\infty\}$ the commutator subgroup is seen to be perfect and we can thus apply the plus construction to the geometric realization of $R_p(A)$.

**Definition 3.1.** [17] *By the* relative $K$-groups *of the unital Banach algebra $A$ we will understand the homotopy groups of the pointed topological space $|R(A)|^+$,*

$$K_n^{\mathrm{rel}}(A) := \pi_n(|R(A)|^+)$$

The relative $K$-groups relate the algebraic $K$-groups and topological $K$-groups through the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r, "i"] & K_{n+1}^{\mathrm{top}}(A) \arrow[r, "v"] & K_n^{\mathrm{rel}}(A) \arrow[r, "\theta"] & K_n(A) \arrow[d, "i"] \\ \cdots & K_{n-1}(A) \arrow[l, "i"'] & K_{n-1}^{\mathrm{rel}}(A) \arrow[l, "\theta"'] & K_n^{\mathrm{top}}(A) \arrow[l, "v"'] \end{tikzcd} \tag{4}$$

Here the map $\theta : K_n^{\mathrm{rel}}(A) \to K_n(A)$ is induced by the map of simplicial sets

$$\theta : R_p(A) \to BGL_p(A) \qquad \sigma \mapsto (\sigma(\mathbf{0})\sigma(\mathbf{1})^{-1}, \ldots \sigma(\mathbf{n-1})\sigma(\mathbf{n})^{-1})$$

For later use we give a result on the homology of the simplicial set $R(A)$. For this, let $R^\infty(A)$ denote the corresponding simplicial set with continuous maps replaced by smooth maps. We then have the following isomorphism,

**Lemma 3.2.** *The simplicial map $i : R^\infty(A) \to R(A)$ given by inclusion, yields an isomorphism in homology, $H_n(R^\infty(A)) \cong H_n(R(A))$.*

*Proof.* For each $r > 0$ we choose a smooth map $\delta_r \in C_c^\infty(\mathbb{R}^n)$ which is supported on the ball around 0 with radius $r$ and which satisfies $\delta_r \geq 0$ and $\int_{\mathbb{R}^n} \delta_r dx = 1$.

For each compactly supported continuous function $\sigma \in C_c(\mathbb{R}^n, M_p(A))$ with $\sigma(t) \in GL_p(A)$ for all $t \in \Delta^n$ we can define the continuous homotopy $H(\sigma) : [0,1] \times \Delta^n \to M_p(A)$

$$H(\sigma)(r,t) = \begin{cases} (\delta_r * \sigma)(t) & r > 0 \\ \sigma(t) & r = 0 \end{cases}$$

Here $*$ denotes the convolution product of compactly supported maps. We note that $H(\sigma)(r, \cdot) : \Delta^n \to GL_p(A)$ for sufficiently small $r > 0$. The result is then a consequence of the $GL_p(A)$-invariance of the homotopy, $H(\sigma \cdot g) = H(\sigma) \cdot g$. $\square$

3.1. **A calculation of the second relative $K$-group.** In this section we show that the second relative $K$-group is isomorphic to the second homology group of a certain simplicial set. The result is thus similar to the result in [18, 22, 23]. Here the second algebraic $K$-group of a ring (using Quillen's definition) is shown to agree with the second homology group of the elementary matrices over the ring. The present calculation is relevant for our explicit definition of the exterior product on the relative $K$-groups.

Let $F(A)_1 \subseteq R(A)_1$ denote the smallest normal subgroup of $R(A)_1$ which contains the commutators

$$\sigma\tau\sigma^{-1}\tau^{-1} \in F(A)_1 \qquad \forall \sigma, \tau \in R(A)_1$$

and which satisfies

$$\alpha \in F(A)_1 \Rightarrow \overline{\alpha} \in F(A)_1 \quad \text{and} \quad \alpha \in F(A)_1 \Rightarrow \big(g\alpha g^{-1} \in F(A)_1 \qquad \forall g \in GL(A)\big)$$

Here $\overline{\alpha} : t \mapsto \alpha(1-t)\cdot\alpha(\mathbf{1})^{-1}$ denotes the inverse path.

Now, we let $\sim_F$ denote the equivalence relation on $R(A)$ which is defined degreewise by

$$x \sim_F y \Leftrightarrow \big(\forall I \in \Delta[n]_{n-1} \exists \tau \in F(A)_1 : d_I(x)\tau = d_I(y)\big)$$

Here $d_I = d_{i_1} \dots d_{i_{n-1}} : R(A)_n \to R(A)_1$ for each $0 \leq i_1 < \dots < i_{n-1} \leq n$. We let $Q(A) = R(A)/\sim_F$ denote the quotient simplicial set.

**Theorem 3.3.** *The quotient map $\pi : R(A) \to Q(A)$ is a Kan fibration.*

*Proof.* Suppose that $x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_n \in R(A)_{n-1}$ are compatible and suppose that $y \in Q(A)_n$ satisfies $d_i(y) = \pi(x_i)$ for all $i \neq k$. We will only consider the problematic case of $n = 2$.

We start by choosing a $z \in R(A)_2$ with $\pi(z) = y$. Suppose that $k = 0$. For each $i \in \{1, 2\}$ we choose a $\tau_i \in F(A)_1$ with $d_i(z)\cdot\tau_i = x_i$. We define the appropriate lift $x \in R(A)_2$ by the formula

$$x(t_1, t_2) = z(t_1, t_2)\tau_1(t_2)\tau_2(t_1) \qquad \forall (t_1, t_2) \in \Delta^2$$

A calculation of the faces of $x \in R(A)_2$ then shows that $\pi(x) = y$, that $d_1(x) = x_1$, and that $d_2(x) = x_2$.

The cases where $k = 1$ or $k = 2$ are treated in a similar fashion. The appropriate lifts are given by

$$x(t_1, t_2) = z(t_1, t_2)\cdot z(1)^{-1}\tau_0(t_2)z(1)\cdot\tau_2(t_1+t_2) \qquad \text{and}$$
$$x(t_1, t_2) = z(t_1, t_2)\cdot z(1)^{-1}\tau_1(1-t_1)\tau_1(1)^{-1}z(1)\cdot\tau_2(t_1+t_2)$$

respectively. □

Let $F(A)$ denote the fiber of the Kan fibration $\pi : R(A) \to Q(A)$

**Lemma 3.4.** *The inclusion $i : F(A) \to R(A)$ induces an isomorphism*

$$\pi_n(i) : \pi_n(F(A)) \to \pi_n(R(A)) \quad \textit{for all} \quad n \geq 2$$

*The fundamental group of the fiber $F(A)$ equals the commutator subgroup of $\pi_1(R(A))$ and the induced map*

$$\pi_1(i) : \big[\pi_1(R(A)), \pi_1(R(A))\big] \to \pi_1(R(A))$$

*is the inclusion.*

*Proof.* We will only consider the calculation of the fundamental group of the fiber. For this, note that

$$\pi_1(F(A)) = F(A)_1/\sim \quad \text{and} \quad \pi_1(R(A)) = R(A)_1/\sim$$

where $\sim$ in both cases denotes homotopies through $GL_p(A)$ with fixed endpoints. The proof of the lemma is then essentially a matter of checking that each element in $\pi_1(F(A))$ can be represented by a commutator of elements in $R(A)_1$. This follows since

$$\overline{\alpha} \sim \alpha^{-1} \quad \text{for each} \quad \alpha \in R(A)_1$$

and since

$$g\alpha g^{-1} \sim \gamma_g \begin{pmatrix} \alpha_{1/2} & 0 \\ 0 & 1 \end{pmatrix} (\gamma_g)^{-1} \quad \text{for each} \quad g \in GL(A)\,,\ \alpha \in R(A)_1$$

Here $\alpha_{1/2}(t) = 1$ for all $t \in [0,1/2)$ and $\gamma_g \in R(A)_1$ satisfies $\gamma_g(t) = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ for all $t \in [1/4, 1]$. □

**Corollary 3.5.** *The homotopy groups of $Q(A)$ are given by*

$$\pi_n(Q(A)) = \begin{cases} 0 & \text{for} \quad n \geq 2 \\ \pi_1(R(A))/[\pi_1(R(A)), \pi_1(R(A))] & \text{for} \quad n = 1 \end{cases}$$

*The induced map $\pi_1(\pi) : \pi_1(R(A)) \to \pi_1(Q(A))$ is the quotient map.*

*Proof.* This is immediate from the long exact sequence of homotopy groups associated with the Kan fibration $F(A) \to R(A) \to Q(A)$ and Lemma 3.4. □

Let $\mathcal{F}$ denote the homotopy fiber of the map $\pi^+ : |R(A)|^+ \to |Q(A)|$ induced by the quotient map $\pi : R(A) \to Q(A)$.

**Theorem 3.6.** *The inclusion $i : F(A) \to R(A)$ gives rise to a homotopy equivalence*

$$f^+ : |F(A)|^+ \to \mathcal{F}$$

*Proof.* This follows from [3] since $\pi_1(Q(A))$ is abelian. □

With this precise description of the homotopy fiber $\mathcal{F}$ in hand, we are able to obtain the desired calculation of the second relative $K$-group.

**Corollary 3.7.** *The space $|F(A)|^+$ is simply connected and the map*

$$i^+ : |F(A)|^+ \to |R(A)|^+$$

*induced by the inclusion $i : F(A) \to R(A)$ yields an isomorphism*

$$\pi_n(i^+) : \pi_n(|F(A)|^+) \to \pi_n(|R(A)|^+) = K_n^{\mathrm{rel}}(A)$$

*for all $n \geq 2$.*

*Proof.* This is immediate from the long exact sequence of homotopy groups arising from the fibration $|F(A)|^+ \to |R(A)|^+ \to |Q(A)|$. □

**Corollary 3.8.** *The second relative $K$-group of the unital Banach algebra $A$ is isomorphic to the second homology group of the simplicial set $F(A)$*

$$K_2^{\mathrm{rel}}(A) \cong H_2(F(A))$$

*Proof.* Since $|F(A)|^+$ is simply connected the Hurewicz homomorphism

$$h_2 : \pi_2(|F(A)|^+) \to H_2(|F(A)|^+) \cong H_2(|F(A)|) \cong H_2(F(A))$$

is an isomorphism. But the group $\pi_2(|F(A)|^+)$ is isomorphic to $K_2^{\mathrm{rel}}(A)$ by Corollary 3.7. □

3.2. **An H-group structure on $|R(A)|^+$.** In this section we show that a pointwise version of the direct sum on $GL(A)$ determines a commutative $H$-group structure on $|R(A)|^+$. Our exposition will follow [18, Section 1.2] and [29] closely.

Let $A$ be a unital Banach algebra. We define the direct sum on the simplicial set $R(A)$ as a pointwise version of the direct sum on $GL(A)$, thus

$$\oplus : R(A) \times R(A) \to R(A) \qquad (\sigma, \tau) \mapsto (t \mapsto \sigma(t) \oplus \tau(t))$$

Let $\oplus^+ : |R(A) \times R(A)|^+ \to |R(A)|^+$ denote the map induced by functoriality of the geometric realization and the plus-construction. Furthermore, let

$$k^{-1} : |R(A)|^+ \times |R(A)|^+ \to |R(A) \times R(A)|^+$$

denote some homotopy inverse of the homotopy equivalence given by the projection onto each factor. We then define the sum on $|R(A)|^+$ as the composition

$$+ = \oplus^+ \circ k^{-1} : |R(A)|^+ \times |R(A)|^+ \to |R(A)|^+$$

This will be the composition in our commutative $H$-group structure on $|R(A)|^+$. The neutral element will be given by the constant map $1 : \Delta^n \to GL(A)$.

Now, to each injection $u : \mathbb{N} \to \mathbb{N}$ there is a group homomorphism $u : GL(A) \to GL(A)$ defined by

$$u(g)_{ij} = \begin{cases} g_{kl} & \text{for} \quad i = u(k), j = u(l) \\ \delta_{ij} & \qquad\qquad \text{elsewhere} \end{cases}$$

We extend this construction to a pointwise version, associating a simplicial map

$$u : R(A) \to R(A) \qquad \sigma \mapsto \big(t \mapsto u(\sigma(t))\big)$$

to each injective map $u : \mathbb{N} \to \mathbb{N}$. We let $u^+ : |R(A)|^+ \to |R(A)|^+$ denote the map induced by functoriality.

**Lemma 3.9.** *For each elementary matrix $g \in E(A)$ the (pre)-simplicial maps*

$$\mathrm{Ad}_g : R(A) \to R(A) \quad \textit{and} \quad \mathrm{Ad}_g : F(A) \to F(A)$$

*given by $\sigma \mapsto g\sigma g^{-1}$ are homotopic to the identity. In particular, for each injection $u : \mathbb{N} \to \mathbb{N}$ we have*

$$u_* = \mathrm{Id} : H_*(R(A)) \to H_*(R(A)) \quad \textit{and} \quad u_* = \mathrm{Id} : H_*(F(A)) \to H_*(F(A))$$

*Proof.* We will only consider the case of $F(A)$. Let $g \in E(A)$. Let $\gamma \in F(A)_1$ satisfy $\gamma(0) = 1$ and $\gamma(1) = g$. We then define a presimplicial homotopy $h_i : F(A)_n \to F(A)_{n+1}$ by

$$h_i(\sigma) = (s_n \dots s_{i+1} s_{i-1} \dots s_0)(\gamma) \cdot s_i(\sigma)$$

proving the first statement of the lemma. To prove the second statement, let $u : \mathbb{N} \to \mathbb{N}$ be injective and note that to each *finite* number of elements $\sigma_1, \dots, \sigma_m \in F(A)_n$ there is an elementary matrix $g \in E(A)$ such that

$$u(\sigma_i) = g \cdot \sigma_i \cdot g^{-1} \quad \text{for all} \quad i \in \{1, \dots, m\}$$

Since each element in $x \in H_n(F(A))$ is represented by a finite number of element in $F(A)_n$ we must have $u_*(x) = x$. □

The results obtained in Section 3.1 together with the classical Whitehead theorem now allows us to show that the monoid of injections $u : \mathbb{N} \to \mathbb{N}$ acts on $|R(A)|^+$ by homotopy equivalences.

**Theorem 3.10.** *For each injection $u : \mathbb{N} \to \mathbb{N}$ the induced map*

$$u^+ : |R(A)|^+ \to |R(A)|^+$$

*is a homotopy equivalence.*

*Proof.* We show that

$$u^+ : |R(A)|^+ \to |R(A)|^+$$

is a weak equivalence and refer to Whitehead's theorem.

For $n = 1$ we note that $\pi_1(|R(A)|^+) \cong H_1(R(A))$, so $\pi_1(u^+) : \pi_1(|R(A)|^+) \to \pi_1(|R(A)|^+)$ is an isomorphism by Lemma 3.9.

For $n \geq 2$ we note that $\pi_n(|F(A)|^+) \cong \pi_n(|R(A)|^+)$ by Corollary 3.7. Since the space $|F(A)|^+$ is simply connected, we will only need to show that $u^+ : |F(A)|^+ \to |F(A)|^+$ induces an isomorphism in homology. However, this is a consequence of Lemma 3.9. □

The commutative $H$-group properties of the sum $+ : |R(A)|^+ \times |R(A)|^+ \to |R(A)|^+$ and the neutral element $1 \in |R(A)|^+$ can now be obtained by a rephrasing of the arguments in [18, Section 1.2].

**Corollary 3.11.** *For each injection $u : \mathbb{N} \to \mathbb{N}$ the induced map*

$$u^+ : |R(A)|^+ \to |R(A)|^+$$

*is homotopic to the identity.*

*Proof.* This is a consequence of the Groethendieck group of the monoid of injections $u : \mathbb{N} \to \mathbb{N}$ being trivial. See [18, Lemma 1.2.8]. □

**Theorem 3.12.** *The application $+ : |R(A)|^+ \times |R(A)|^+ \to |R(A)|^+$ and the neutral element $1 \in |R(A)|^+$ define a commutative $H$-group structure on $|R(A)|^+$.*

*Proof.* That the sum and the neutral element defines a homotopy associative and homotopy commutative $H$-space structure follows from Corollary 3.11 since the appropriate maps are homotopic up to composition with some $u^+ : |R(A)|^+ \to |R(A)|^+$ . The existence of a homotopy inverse is automatic since we are working exclusively with connected $CW$-complexes, [25, Theorem 3.4]. □

We end this section by showing that the map $\theta : |R(A)|^+ \to BGL(A)^+$, induced by the simplicial map $\theta : \sigma \mapsto (\sigma(\mathbf{0})\sigma(\mathbf{1})^{-1}, \ldots, \sigma(\mathbf{n-1})\sigma(\mathbf{n})^{-1})$, respects the $H$-group structures.

**Theorem 3.13.** *The map $\theta : |R(A)|^+ \to BGL(A)^+$ is an $H$-map.*

*Proof.* This is essentially a matter of checking that the simplicial maps given by

$$\theta \circ \oplus \text{ and } \oplus \circ (\theta \times \theta) : R(A) \times R(A) \to BGL(A)$$

coincide. □

3.3. **Construction of the product in relative $K$-theory.** In this section we show that a pointwise version of the exterior Loday product, determines a multiplicative structure on the relative $K$-groups. The exposition will follow [18, Section 2.1] closely.

Let $A$ and $B$ be unital Banach algebras.

Let $p, q \in \{3, 4, \ldots\}$ be fixed and let $\varphi : A^p \otimes_{\mathbb{Z}} B^q \to (A \otimes_{\mathbb{Z}} B)^{pq}$ denote some isomorphism of $A \otimes_{\mathbb{Z}} B$-bimodules. As in Section 2.1 we have a corresponding map of simplicial sets

$$\hat{\otimes}_\varphi : R_p(A) \times R_q(B) \to R_{pq}(A \hat{\otimes} B)$$

We let $\hat{\otimes}_\varphi^+ : |R_p(A) \times R_q(B)|^+ \to |R_{pq}(A \hat{\otimes} B)|$ denote the induced map between the plus-constructions. Furthermore, let

$$k^{-1} : |R_p(A)|^+ \times |R_q(B)|^+ \to |R_p(A) \times R_q(B)|^+$$

denote some homotopy inverse to the map given by the projection onto each factor. We then define the tensor product

$$\hat{\otimes}^+ = \iota^+ \circ \hat{\otimes}_\varphi^+ \circ k^{-1} : |R_p(A)|^+ \times |R_q(B)|^+ \to |R(A \hat{\otimes} B)|^+$$

Here $\iota^+ : |R_{pq}(A \hat{\otimes} B)|^+ \to |R(A \hat{\otimes} B)|^+$ is determined by functoriality from the inclusion $\iota : R_{pq}(A \hat{\otimes} B) \to R(A \hat{\otimes} B)$.

Following the argumentation of Theorem 3.12 we see that the tensor product thus defined is natural in $A$ and $B$, bilinear, associative and commutative up to homotopy. Furthermore, it only depends on the choice of isomorphism $\varphi : A^p \otimes_{\mathbb{Z}} B^q \to (A \otimes_{\mathbb{Z}} B)^{pq}$ of $(A \otimes_{\mathbb{Z}} B)$-bimodules up to homotopy. See also [18, Section 2.1.2].

Now, in order to get a map which descends to the smash product and which behaves well when $p$ and $q$ tend to infinity we define

$$\gamma_{p,q}^{\text{rel}} : |R_p(A)|^+ \times |R_q(B)|^+ \to |R(A \hat{\otimes} B)|^+$$
$$\gamma_{p,q}^{\text{rel}} : (x, y) \mapsto x \hat{\otimes}^+ y - x \hat{\otimes}^+ 1_q - 1_p \hat{\otimes}^+ y + 1_p \hat{\otimes}^+ 1_q$$

Here the minus sign comes from the (commutative) $H$-group structure on $|R(A\hat{\otimes}B)|^+$ defined in Section 3.2. The elements $1_p \in |R_p(A)|^+$ and $1_q \in |R_q(B)|^+$ are given by the constant maps $1_p : \Delta^n \to GL_p(A)$ and $1_q : \Delta^n \to GL_q(B)$.

It is then immediate that the restriction $\gamma_{p,q}^{\mathrm{rel}} : |R_p(A)|^+ \vee |R_q(B)|^+ \to |R(A\hat{\otimes}B)|^+$ is homotopically trivial. Since $|R(A\hat{\otimes}B)|^+$ is an $H$-group we thus get a map on the smash product

$$\widehat{\gamma}_{p,q}^{\mathrm{rel}} : |R_p(A)|^+ \wedge |R_q(B)|^+ \to |R(A\hat{\otimes}B)|^+$$

which is unique up to homotopy and which makes the maps

$$\widehat{\gamma}_{p,q}^{\mathrm{rel}} \circ \pi \text{ and } \gamma_{p,q}^{\mathrm{rel}} : |R_p(A)|^+ \times |R_q(B)|^+ \to |R(A\hat{\otimes}B)|^+$$

homotopic. Here $\pi : |R_p(A)|^+ \times |R_q(B)|^+ \to |R_p(A)|^+ \wedge |R_q(B)|^+$ denotes the quotient map.

The argumentation explicited in [18, p.333-335] now ensures the existence of a continuous map

$$\widehat{\gamma}^{\mathrm{rel}} : |R(A)|^+ \wedge |R(B)|^+ \to |R(A\hat{\otimes}B)|^+$$

which is natural in $A$ and $B$, bilinear, associative and commutative up to weak homotopies. Furthermore, for any $p, q \in \{3, 4, \ldots\}$ the maps

$$\widehat{\gamma}_{p,q}^{\mathrm{rel}} \text{ and } \widehat{\gamma}^{\mathrm{rel}} \circ (\iota \wedge \iota) : |R_p(A)|^+ \wedge |R_q(B)|^+ \to |R(A\hat{\otimes}B)|^+$$

agree up to weak homotopy. Here $\iota \wedge \iota : |R_p(A)|^+ \wedge |R_q(B)|^+ \to |R(A)|^+ \wedge |R(B)|^+$ denotes the inclusion. This enables us to make the following definition.

**Definition 3.14.** *By the* exterior product in relative $K$-theory *we understand the map*

$$*^{\mathrm{rel}} : K_n^{\mathrm{rel}}(A) \times K_m^{\mathrm{rel}}(B) \to K_{n+m}^{\mathrm{rel}}(A\hat{\otimes}B)$$

*given by the formula*

$$[f] *^{\mathrm{rel}} [g] = [\widehat{\gamma}^{\mathrm{rel}} \circ (f \wedge g)]$$

*for each* $[f] \in \pi_n(|R(A)|^+)$ *and* $[g] \in \pi_m(|R(B)|^+)$.

The naturality, bilinearity and associativity up to weak homotopies of the map $\widehat{\gamma}^{\mathrm{rel}} : |R(A)|^+ \wedge |R(B)|^+ \to |R(A\hat{\otimes}B)|^+$ imply the corresponding properties for the exterior product.

**Theorem 3.15.** *The exterior product in relative $K$-theory*

$$*^{\mathrm{rel}} : K_n^{\mathrm{rel}}(A) \times K_m^{\mathrm{rel}}(B) \to K_{n+m}^{\mathrm{rel}}(A\hat{\otimes}B)$$

*is natural, bilinear and associative.*

In the case where the unital Banach algebra $A$ is commutative we get an interior product

$$*^{\mathrm{rel}} : K_n^{\mathrm{rel}}(A) \times K_m^{\mathrm{rel}}(A) \to K_{n+m}^{\mathrm{rel}}(A)$$

given by composition of the exterior product with the map induced by the continuous algebra homomorphism $\nabla : A\hat{\otimes}A \to A$ , $a_1 \otimes a_2 \mapsto a_1 \cdot a_2$. We are thus able to equip the direct sum of relative $K$-groups $\oplus_{n=1}^{\infty} K_n^{\mathrm{rel}}(A)$ with the structure of a graded commutative ring. See also [18, Theorem 2.1.12].

3.4. **Relations with the Loday product.** Our task is now to compare the exterior product of Loday in algebraic $K$-theory with the exterior product in relative $K$-theory.

Let $A$ and $B$ be unital Banach algebras. Recall that the Loday product

$$* : K_n(A) \times K_m(B) \to K_{n+m}(A \otimes_{\mathbb{Z}} B)$$

is uniquely determined by the continuous maps

$$\begin{aligned} \gamma_{p,q} : BGL_p(A)^+ \times BGL_q(B)^+ &\to BGL(A \otimes_{\mathbb{Z}} B)^+ \\ (x, y) &\mapsto x \otimes^+ y - 1_p \otimes^+ y - x \otimes^+ 1_q + 1_p \otimes^+ 1_q \end{aligned}$$

Here the tensorproduct $\otimes^+ : BGL_p(A)^+ \times BGL_q(B)^+ \to BGL(A \otimes_{\mathbb{Z}} B)^+$ is induced by the group homomorphism

$$\otimes_\varphi : GL_p(A) \times GL_q(B) \to GL_{pq}(A \otimes_{\mathbb{Z}} B) \subseteq GL(A \otimes_{\mathbb{Z}} B)$$

associated with an isomorphism $\varphi : A^p \otimes_{\mathbb{Z}} B^q \to (A \otimes_{\mathbb{Z}} B)^{pq}$ of $A \otimes_{\mathbb{Z}} B$-bimodules. The additive compositions come from the (commutative) $H$-group structure on $BGL(A \otimes_{\mathbb{Z}} B)^+$, [18].

**Definition 3.16.** *By the* completed Loday product *in algebraic $K$-theory we will understand the composition*

$$\hat{*} = \iota_* \circ * : K_n(A) \times K_m(B) \to K_{n+m}(A \hat{\otimes} B)$$

*of the Loday product and the map induced by the "identity" ring homomorphism $\iota : A \otimes_{\mathbb{Z}} B \to A \hat{\otimes} B$*

We can then show that the homomorphism $\theta : K_n^{\mathrm{rel}}(A) \to K_n(A)$ respects the exterior product structures.

**Theorem 3.17.** *For each $x \in K_n^{\mathrm{rel}}(A)$ and each $y \in K_m^{\mathrm{rel}}(B)$ we have the equality*

$$\theta(x *^{\mathrm{rel}} y) = \theta(x) \,\hat{*}\, \theta(y)$$

*in $K_{n+m}(A \hat{\otimes} B)$. In particular, the map $\theta : \oplus_{n \geq 1} K_n^{\mathrm{rel}}(A) \to \oplus_{n \geq 1} K_n(A)$ is a homomorphism of graded commutative rings whenever $A$ is a commutative, unital Banach algebra.*

*Proof.* Let $p, q \in \{3, 4, \ldots\}$ and let $\varphi : A^p \otimes_{\mathbb{Z}} B^q \to (A \otimes_{\mathbb{Z}} B)^{pq}$ denote some isomorphism of $(A \otimes_{\mathbb{Z}} B)$-bimodules. On the level of simplicial sets we then have the equality

$$\theta(\sigma \hat{\otimes}_\varphi \tau) = \iota(\theta(\sigma) \otimes_\varphi \theta(\tau)) \quad \text{for all} \quad \sigma \in R_p(A)_n \,,\, \tau \in R_q(B)_n$$

This shows that the maps

$$\theta \circ \hat{\otimes}^+ \ \text{ and } \ \iota^+ \circ \otimes^+ \circ (\theta \times \theta) : |R_p(A)|^+ \times |R_q(B)|^+ \to BGL(A \hat{\otimes} B)^+$$

are homotopic. By Theorem 3.13 the map $\theta : |R(A \hat{\otimes} B)|^+ \to BGL(A \hat{\otimes} B)^+$ respects the $H$-group structures up to homotopy so the maps

$$\theta \circ \gamma_{p,q}^{\mathrm{rel}} \ \text{ and } \ \iota^+ \circ \gamma_{p,q} \circ (\theta \times \theta) : |R_p(A)|^+ \times |R_q(B)|^+ \to BGL(A \hat{\otimes} B)^+$$

are homotopic. The desired result now follows by uniqueness of the involved constructions. $\square$

## 4. On the multiplicative properties of the relative Chern character

Let $A$ be a unital Banach algebra. Let us start by recalling the construction of the relative Chern character as introduced by A. Connes and M. Karoubi, [12, 17]. By definition, the relative Chern character is obtained as the composition of four maps

$$\mathrm{ch}^{\mathrm{rel}} : K_n^{\mathrm{rel}}(A) \to HC_{n-1}(A) \qquad \mathrm{ch}^{\mathrm{rel}} = \mathrm{TR} \circ \varepsilon \circ L \circ h_n$$

We will give a brief description of each of the maps.

The first map is the Hurewicz homomorphism associated with the pointed topological space $|R(A)|^+$,

$$h_n : K_n^{\mathrm{rel}}(A) = \pi_n(|R(A)|^+) \to H_n(|R(A)|^+) \cong H_n(R(A))$$

The second map is the logarithm

$$L : H_n(R(A)) \cong H_n(R^\infty(A)) \to \lim_{p\to\infty} H_n^{\mathrm{Lie}}(M_p(A))$$

which is given by the chain map

$$L : \sigma \mapsto \int_{\Delta^n} \frac{\partial \sigma}{\partial t_1} \cdot \sigma^{-1} \wedge \ldots \wedge \frac{\partial \sigma}{\partial t_n} \cdot \sigma^{-1} dt_1 \ldots dt_n \tag{5}$$

Here $\sigma : \Delta^n \to GL_p(A)$ is a smooth function. See [26]. The isomorphism $H_n(R(A)) \cong H_n(R^\infty(A))$ was proved in Lemma 3.2. Note that we are working with the continuous Lie algebra complex $(\Lambda_*(A), \delta)$, thus in each degree we have a Banach space, $\Lambda_n(A)$, in the appropriate quotient norm. This is needed in order for the above integral to make sense. See also Section 2.2.

The third map is the antisymmetrization

$$\varepsilon : \lim_{p\to\infty} H_n^{\mathrm{Lie}}(M_p(A)) \to \lim_{p\to\infty} HC_{n-1}(M_p(A))$$

which is given by the continuous map

$$\varepsilon : x_0 \wedge x_1 \wedge \ldots \wedge x_{n-1} \mapsto \sum_{s\in\Sigma_{n-1}} \mathrm{sgn}(s) x_0 \otimes x_{s(1)} \otimes \ldots \otimes x_{s(n-1)}$$

Again, we are working with the continuous cyclic complex $(C_n^\lambda(A), b)$, thus in each degree we have a Banach space, $C_n^\lambda(A)$, in the appropriate quotient norm. See [19, 20].

The last map is the generalized trace on continuous cyclic homology

$$\mathrm{TR} : \lim_{p\to\infty} HC_{n-1}(M_p(A)) \to HC_{n-1}(A)$$

See [19] for example.

By [12, Theorem 3.7] the relative Chern character fits in the (up to constants) commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "i"'] & K_{n+1}^{\mathrm{top}}(A) \arrow[r, "v"] \arrow[d, "\mathrm{ch}_{n+1}^{\mathrm{top}}"'] & K_n^{\mathrm{rel}}(A) \arrow[r, "\theta"] \arrow[d, "\mathrm{ch}_n^{\mathrm{rel}}"'] & K_n(A) \arrow[r, "i"] \arrow[d, "D_n"'] & K_n^{\mathrm{top}}(A) \arrow[r, "v"] \arrow[d, "\mathrm{ch}_n^{\mathrm{top}}"'] & \cdots \\
\cdots \arrow[r, "I"'] & HC_{n+1}(A) \arrow[r, "S"'] & HC_{n-1}(A) \arrow[r, "B"'] & HH_n(A) \arrow[r, "I"'] & HC_n(A) \arrow[r, "S"'] & \cdots
\end{tikzcd}$$

Here the other columns are the Dennis trace and the topological Chern character. The bottom row is the $SBI$-sequence in continous homology. Remark that the relative Chern character defined in this section differs from the one given in [12, 17] by the constant $(-1)^n(n-1)!$ on $K_n^{\mathrm{rel}}(A)$. This is necessary to make the map respect the product structures.

4.1. **The multiplicative properties of the logarithm.** Let $A$ and $B$ be unital Banach algebras. In this section we will show that the logarithm $L : H_*(R_p(A)) \to H_*^{\mathrm{Lie}}(M_p(A))$ respects the product structures on the homology of the simplicial sets $R_p(A)$ and the Lie algebra homology of the Banach algebras $M_p(A)$. These exterior products were introduced in Section 2.1 and Section 2.2.

For each $n, p \in \mathbb{N}$ and each $j \in \{1, \dots, n\}$ we define the operator

$$\Gamma_j : C^\infty(\Delta^n, GL_p(A)) \to C^\infty(\Delta^n, M_p(A)) \qquad \Gamma_j : \sigma \mapsto \tfrac{\partial \sigma}{\partial t_j} \cdot \sigma^{-1}$$

Furthermore we define the wedge product

$$\gamma : C^\infty(\Delta^n, GL_p(A)) \to C^\infty(\Delta^n, \Lambda_n(M_p(A))) \qquad \gamma(\sigma)(t) = \Gamma_1(\sigma)(t) \wedge \dots \wedge \Gamma_n(\sigma)(t)$$

Our first task is then to understand the behaviour of $\gamma$ with respect to the exterior shuffle product. This is the content of Lemma 4.1. However we will start by introducing some convenient notation.

Let us fix to smooth maps $\sigma : \Delta^n \to GL_p(A)$ and $\tau : \Delta^m \to GL_q(B)$ and let us choose an isomorphism $\varphi : A^p \otimes_{\mathbb{Z}} B^q \to (A \otimes_{\mathbb{Z}} B)^{pq}$ of $(A \otimes_{\mathbb{Z}} B)$-bimodules. Furthermore we let $(\mu, \nu) \in \Sigma_{(n,m)}$ be a fixed $(n,m)$-shuffle. To ease the exposition we will assume that $\mu(0) = 0$ and that $\nu(m-1) = n + m - 1$.

Let $\{A_0, A_1, \dots, A_{2k+1}\}$ denote the unique partition of $\{0, 1, \dots, n+m-1\}$ satisfying the conditions

$$\textstyle\bigcup_{i=0}^k A_{2i} = \mathrm{Im}(\mu)\,, \qquad \bigcup_{i=0}^k A_{2i+1} = \mathrm{Im}(\nu) \quad \text{and} \quad i < j \Rightarrow (x < y \qquad \forall x \in A_i\,,\, y \in A_j)$$

Let $k_i$ denote the smallest element in $A_i$ and let $k_{2k+2} = n + m$. We associate the composition of degeneracies $s_{A_i} = s_{k_{i+1}-1} \dots s_{k_i}$ to each set $A_i$ in the partition. We then have the equality

$$s_\nu(\sigma) \hat{\otimes}_\varphi s_\mu(\tau) = s_{A_{2k+1}} \dots s_{A_3} s_{A_1}(\sigma) \hat{\otimes}_\varphi s_{A_{2k}} \dots s_{A_2} s_{A_0}(\tau) : \Delta^{n+m} \to GL_{pq}(A \hat{\otimes} B)$$

For each $j \in \{0, \dots, k\}$ we let

$$E_j = \textstyle\sum_{i=0}^j |A_{2i}| = \sum_{i=0}^j (k_{2i+1} - k_{2i}) \quad \text{and} \quad O_j = \sum_{i=0}^j |A_{2i+1}| = \sum_{i=0}^j (k_{2i+2} - k_{2i+1})$$

We then define the smooth maps

$$\omega_{2j} : \Delta^{n+m} \to \Lambda_{|A_{2j}|} M_{pq}(A \hat{\otimes} B) \quad \text{and} \quad \omega_{2j+1} : \Delta^{n+m} \to \Lambda_{|A_{2j+1}|} M_{pq}(A \hat{\otimes} B)$$

by the wedge products

$$\begin{aligned} \omega_{2j} &= s_\nu(\Gamma_{E_{j-1}+1}(\sigma \hat{\otimes}_\varphi 1_q) \wedge \dots \wedge \Gamma_{E_j}(\sigma \hat{\otimes}_\varphi 1_q)) \qquad \text{and} \\ \omega_{2j+1} &= s_\mu(\Gamma_{O_{j-1}+1}(1_p \hat{\otimes}_\varphi \tau) \wedge \dots \wedge \Gamma_{O_j}(1_p \hat{\otimes}_\varphi \tau)) \end{aligned}$$

Finally, let us recall the relations between the degeneracies and the partial differential operators,

$$(6) \qquad \frac{\partial}{\partial t_j} \circ s_i = \begin{cases} s_i \circ \frac{\partial}{\partial t_{j-1}} & \text{for} \quad j > i > 0 \\ s_i \circ \frac{\partial}{\partial t_j} & \text{for} \quad j \leq i \end{cases} \qquad \frac{\partial}{\partial t_j} \circ s_0 = \begin{cases} s_0 \circ \frac{\partial}{\partial t_{j-1}} & \text{for} \quad j > 1 \\ 0 & \text{for} \quad j = 1 \end{cases}$$

We can then prove the following technical result,

**Lemma 4.1.** *Let $\sigma : \Delta^n \to GL_p(A)$ and $\tau : \Delta^m \to GL_q(B)$ be a pair of smooth maps. For each $(n,m)$-shuffle $(\mu,\nu) \in \Sigma_{(n,m)}$ we then have the equality*

$$\gamma(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau)) = \text{sgn}(\mu,\nu) s_\nu(\gamma(\sigma\hat{\otimes}_\varphi 1_q)) \wedge s_\mu(\gamma(1_p\hat{\otimes}_\varphi \tau))$$

*between smooth maps $\Delta^{n+m} \to \Lambda_{n+m} M_{pq}(A\hat{\otimes}B)$*

*Proof.* We will assume that $\mu(0) = 0$ and that $\nu(m-1) = n+m-1$. The other cases can be proved using similar arguments.

We start by noting that

$$\begin{aligned} \omega_0 \wedge \omega_1 \wedge \ldots \wedge \omega_{2k+1} &= \text{sgn}(\mu,\nu)(\omega_0 \wedge \omega_2 \wedge \ldots \wedge \omega_{2k}) \wedge (\omega_1 \wedge \omega_3 \wedge \ldots \wedge \omega_{2k+1}) \\ &= \text{sgn}(\mu,\nu) s_\nu(\gamma(\sigma\hat{\otimes}_\varphi 1_q)) \wedge s_\mu(\gamma(1_p\hat{\otimes}_\varphi \tau)) \end{aligned}$$

It is therefore sufficient to prove the identity

$$\gamma(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau)) = \omega_0 \wedge \ldots \wedge \omega_{2k+1}$$

We will use induction to show that

$$\Gamma_1(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau)) \wedge \ldots \wedge \Gamma_{k_i}(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau)) = \omega_0 \wedge \ldots \wedge \omega_{i-1}$$

for each $i \in \{1, \ldots, 2k+2\}$. Thus, let $j \in \{1, \ldots, k_1\}$. By the identities in (6) we get

$$\begin{aligned} \frac{\partial}{\partial t_j}(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau)) &= \frac{\partial}{\partial t_j}(s_\nu(\sigma))\hat{\otimes}_\varphi s_\mu(\tau) + s_\nu(\sigma)\hat{\otimes}_\varphi \frac{\partial}{\partial t_j}(s_\mu(\tau)) \\ &= s_\nu(\frac{\partial \sigma}{\partial t_j})\hat{\otimes}_\varphi s_\mu(\tau) \end{aligned}$$

By consequence we have that

$$\Gamma_j(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau)) = s_\nu(\Gamma_j(\sigma))\hat{\otimes}_\varphi 1_q = s_\nu(\Gamma_j(\sigma\hat{\otimes}_\varphi 1_q))$$

proving the induction start.

Now, suppose that

$$\Gamma_1(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau)) \wedge \ldots \wedge \Gamma_{k_i}(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau)) = \omega_0 \wedge \ldots \wedge \omega_{i-1}$$

for some $i \in \{1, \ldots, 2k+1\}$. We will only consider the case of $i = 2r$ being even. The odd case can be proven by similar arguments. Thus, let $j \in \{k_{2r}+1, \ldots, k_{2r+1}\}$. By the identities in (6)

we get

$$\begin{aligned}\frac{\partial}{\partial t_j}(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau)) &= s_{A_{2k+1}}\ldots s_{A_{2r+1}}\frac{\partial}{\partial t_j}(s_{A_{2r-1}}\ldots s_{A_1}(\sigma))\hat{\otimes}_\varphi s_\mu(\tau)\\ &\quad + s_\nu(\sigma)\hat{\otimes}_\varphi s_{A_{2k}}\ldots s_{A_{2r+2}}\frac{\partial}{\partial t_j}(s_{A_{2r}}\ldots s_{A_0}(\tau))\\ &= s_\nu(\frac{\partial\sigma}{\partial t_{j-O_{r-1}}})\hat{\otimes}_\varphi s_\mu(\tau) + s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\frac{\partial\tau}{\partial t_{k_{2r}-E_{r-1}}})\end{aligned}$$

Noting that $k_{2r} - E_{r-1} = O_{r-1}$ we deduce the identity

$$\Gamma_j(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau)) = s_\nu(\Gamma_{j-O_{r-1}}(\sigma\hat{\otimes}_\varphi 1_q)) + s_\mu(\Gamma_{O_{r-1}}(1_p\hat{\otimes}_\varphi\tau))$$

But the term $s_\mu(\Gamma_{O_{r-1}}(1_p\hat{\otimes}_\varphi\tau))$ already appears in the wedge product

$$\omega_{2r-1} = s_\mu(\Gamma_{O_{r-2}+1}(1_p\hat{\otimes}_\varphi\tau)\wedge\ldots\wedge\Gamma_{O_{r-1}}(1_p\hat{\otimes}_\varphi\tau))$$

Using the induction hypothesis we thus get that

$$\begin{aligned}&\Gamma_1(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau))\wedge\ldots\wedge\Gamma_{k_{2r+1}}(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau))\\ &\quad= \big(\omega_0\wedge\ldots\wedge\omega_{2r-1}\big)\wedge s_\nu\big(\Gamma_{k_{2r}+1-O_{r-1}}(\sigma\hat{\otimes}_\varphi 1_q)\wedge\ldots\wedge\Gamma_{k_{2r+1}-O_{r-1}}(\sigma\hat{\otimes}_\varphi 1_q)\big)\\ &\quad= \omega_0\wedge\ldots\wedge\omega_{2r}\end{aligned}$$

proving the induction step. □

To continue further, we will need the following Lemma which can be proved by a direct but tedious computation,

**Lemma 4.2.** *For any pair of continuous maps $\alpha : \Delta^n \to A$ and $\beta : \Delta^m \to B$ we have the identity*

$$\begin{aligned}&\sum_{(\mu,\nu)\in\Sigma_{(n,m)}}\int_{\Delta^{n+m}} s_\nu(\alpha)\otimes s_\mu(\beta)\,dt_1\ldots dt_{n+m}\\ &\qquad= (\int_{\Delta^n}\alpha\,dt_1\ldots dt_n)\otimes(\int_{\Delta^m}\beta\,dt_1\ldots dt_m)\end{aligned}$$

*in the unital Banach algebra $A\hat{\otimes}B$.*

Let $\phi : M_p(A)\hat{\otimes}M_q(B)\to M_{pq}(A\hat{\otimes}B)$ denote the continuous algebra homomorphism associated with the choice of the isomorphism $\varphi : A^p\otimes_{\mathbb{Z}} B^q \to (A\otimes_{\mathbb{Z}} B)^{pq}$ of $(A\otimes_{\mathbb{Z}} B)$-bimodules. We are now ready for the main result of this section,

**Theorem 4.3.** *For each pair of smooth maps $\sigma : \Delta^n \to GL_p(A)$ and $\tau : \Delta^m \to GL_q(B)$ we have the equality*

$$L(\sigma\times_\varphi\tau) = \phi_*(L(\sigma)\wedge^E L(\tau))$$

*in $\Lambda_{n+m}M_{pq}(A\hat{\otimes}B)$.*

*Proof.* Using Lemma 4.1 and Lemma 4.2 we get that

$$\begin{aligned}
L(\sigma \times_\varphi \tau) &= \sum_{(\mu,\nu)\in\Sigma_{(n,m)}} \operatorname{sgn}(\mu,\nu) \int_{\Delta^{n+m}} \gamma(s_\nu(\sigma)\hat{\otimes}_\varphi s_\mu(\tau))\, dt_1 \dots dt_{n+m} \\
&= \sum_{(\mu,\nu)\in\Sigma_{(n,m)}} \int_{\Delta^{n+m}} s_\nu\big(\gamma(\sigma\hat{\otimes}_\varphi 1_q)\big) \wedge s_\mu\big(\gamma(1_p\hat{\otimes}_\varphi \tau)\big)\, dt_1 \dots dt_{n+m} \\
&= L(\sigma\hat{\otimes}_\varphi 1_q) \wedge L(1_p\hat{\otimes}_\varphi \tau)
\end{aligned}$$

The desired result now follows by naturality of the logarithm. □

4.2. **The multiplicative properties of the antisymmetrization.** Let $A$ and $B$ be unital Banach algebras. We will now show that the antisymmetrization $\varepsilon : \Lambda_* A \to C^\lambda_{*-1}(A)$ respects the product structures on the continuous Lie algebra homology and the continuous cyclic homology. The definition of the antisymmetrization is recalled in the beginning of this section and the exterior products considered are defined in Section 2.2 and Section 2.3.

**Theorem 4.4.** *For each $x \in \Lambda_n A$ and each $y \in \Lambda_m B$ we have the equality*

$$\varepsilon(x \wedge^E y) = \varepsilon(x) * \varepsilon(y)$$

*in $C^\lambda_{n+m-1}(A\hat{\otimes}B)$.*

*Proof.* Let $x = x_0 \wedge x_1 \wedge \dots \wedge x_{n-1} \in \Lambda_n A$ and let $y = y_0 \wedge y_1 \wedge \dots \wedge y_{m-1} \in \Lambda_m B$. For each $i \in \{0, 1, \dots, n+m-1\}$, let

$$z_i = \begin{cases} x_i \otimes 1_B & \text{for} \quad i \in \{0, \dots, n-1\} \\ 1_A \otimes y_{i-n} & \text{for} \quad i \in \{n, \dots, n+m-1\} \end{cases}$$

By definition of the exterior wedge product and the antisymmetrization map we get

$$\varepsilon(x \wedge^E y) = \sum_{s\in\Sigma_{n+m-1}} \operatorname{sgn}(s) z_0 \otimes z_{s(1)} \otimes \dots \otimes z_{s(n+m-1)}$$

However, using the bijective correspondence

$$\Sigma_{(n-1,m)} \times (\Sigma_{n-1} \times \Sigma_m) \to \Sigma_{n+m-1} \qquad (\mu, (\sigma \times \tau)) \mapsto \mu \circ (\sigma \times \tau)$$

we recognize the right hand side as the exterior Hochschild shuffle product of the elements

$$\textstyle\sum_{\sigma\in\Sigma_{n-1}} \operatorname{sgn}(\sigma) x_0 \otimes x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n-1)} \quad \text{and} \quad \sum_{\tau\in\Sigma_m} \operatorname{sgn}(\tau) 1_B \otimes y_{\tau(0)} \otimes \dots \otimes y_{\tau(m-1)}$$

We therefore have

$$\varepsilon(x \wedge^E y) = \varepsilon(x) \times \varepsilon(1_B \wedge y) = \varepsilon(x) \times (sN\varepsilon)(y) = \varepsilon(x) * \varepsilon(y)$$

proving the desired result. □

4.3. **The multiplicative properties of the generalized trace.** In this section we will show that the generalized trace $\mathrm{TR} : C^\lambda_*(M_p(A)) \to C^\lambda_*(A)$ respects the exterior product of degree one in continuous cyclic homology.

Let $\phi : M_p(A)\hat{\otimes}M_q(B) \to M_{pq}(A\hat{\otimes}B)$ denote the continuous algebra homomorphism induced by some isomorphism $\varphi : A^p \otimes_{\mathbb{Z}} B^q \to (A \otimes_{\mathbb{Z}} B)^{pq}$ of $(A \otimes_{\mathbb{Z}} B)$-bimodules.

**Theorem 4.5.** *For each $x \in C^\lambda_n(M_p(A))$ and each $y \in C^\lambda_m(M_q(B))$ we have the equality*

$$\mathrm{TR}(x) * \mathrm{TR}(y) = (\mathrm{TR} \circ \phi_*)(x * y)$$

*in $C^\lambda_{n+m+1}(A\hat{\otimes}B)$.*

*Proof.* Let $u \in M_p(\mathbb{C})$ and let $v \in M_q(\mathbb{C})$. We start by noticing the identity $\mathrm{Tr}(\phi(u \otimes v)) = \mathrm{Tr}(u)\mathrm{Tr}(v)$. Here $\mathrm{Tr} : M_k(\mathbb{C}) \to \mathbb{C}$ denotes the usual trace.

Using the formula of [19, Lemma 1.2.2] for the generalized trace we thus get that

$$(\mathrm{TR} \circ \phi_*)(x \times (sN)(y)) = \mathrm{TR}(x) \times (\mathrm{TR}sN)(y)$$

The result of the Theorem then follows from the identity $(\mathrm{TR}sN)y) = (sN\mathrm{TR})(y)$, see [19, Lemma 2.2.8] for example. □

4.4. **The multiplicative properties of the Hurewicz homomorphism.** Let $A$ and $B$ be unital Banach algebras. In this section we will investigate the behaviour of the Hurewicz homomorphism with respect to the exterior product in relative $K$-theory and the exterior shuffle product on the homology of the simplicial sets $R_p(A)$. The exterior product in relative $K$-theory was constructed in Section 3 and the exterior shuffle product was defined in Section 2.1.

For each $n \in \mathbb{N}$ denote the class of $1 \in \mathbb{Z}$ under the isomorphism $\mathbb{Z} \cong H_n(S^n)$ by $\mathbf{1}_n \in H_n(S^n)$. Furthermore, we let

$$\mathrm{sh} : H_n(S^n) \otimes_{\mathbb{Z}} H_m(S^m) \to H_{n+m}(S^n \times S^m)$$

denote the shuffle map in singular homology. Let $\pi : S^n \times S^m \to S^n \wedge S^m$ denote the quotient map. We then get the equality

$$(\pi_* \circ \mathrm{sh})(\mathbf{1}_n \otimes \mathbf{1}_m) = \mathbf{1}_{n+m} \tag{7}$$

in $H_{n+m}(S^{n+m}) \cong \mathbb{Z}$. For notational reasons we define

$$\zeta := \mathrm{sh}(\mathbf{1}_n \otimes \mathbf{1}_m) \in H_{n+m}(S^n \times S^m)$$

The next Lemma is the first step needed in order to express the Hurewicz homomorphism of a product in terms of the Hurewicz homomorphism of the original elements.

**Lemma 4.6.** *Let $p, q \in \{3, 4, \dots\}$. Let $f : S^n \to |R_p(A)|^+$ and $g : S^m \to |R_q(B)|^+$ be continuous maps. We then have the equality*

$$(f\hat{\otimes}^+ g)_*(\zeta) = \iota_*(h_n(f) \times h_m(g))$$

*in $H_{n+m}(R(A\hat{\otimes}B)) \cong H_{n+m}(|R(A\hat{\otimes}B)|^+)$. Here $\iota : R_{pq}(A\hat{\otimes}B) \to R(A\hat{\otimes}B)$ denotes the inclusion.*

*Proof.* Let us fix an isomorphism $\varphi : A^p \otimes_{\mathbb{Z}} B^q \to (A \otimes_{\mathbb{Z}} B)^{pq}$ of $(A \otimes_{\mathbb{Z}} B)$-bimodules.

Up to canonical identifications in homology we get that the compositions

$$\begin{aligned} \hat{\otimes}_*^+ \circ \mathrm{sh} : & H_n(|R_p(A)|^+) \otimes_{\mathbb{Z}} H_m(|R_q(B)|^+) \\ & \to H_{n+m}(|R_p(A)|^+ \times |R_q(B)|^+) \to H_{n+m}(|R(A \hat{\otimes} B)|^+) \qquad \text{and} \\ \iota_* \circ (\hat{\otimes}_\varphi)_* \circ \mathrm{sh} : & H_n(R_p(A)) \otimes_{\mathbb{Z}} H_m(R_q(B)) \\ & \to H_{n+m}(R_p(A) \times R_q(B)) \to H_{n+m}(R(A \hat{\otimes} B)) \end{aligned}$$

coincide. See Section 2.1 and Section 3.3.

By definition of the exterior shuffle product and the Hurewicz homomorphism we thus have

$$\iota_*(h_n(f) \times h_m(g)) = (\hat{\otimes}_*^+ \circ \mathrm{sh})(f_*(\mathbf{1}_n) \otimes g_*(\mathbf{1}_m)) = (\hat{\otimes}_*^+ \circ (f \times g)_*)(\zeta)$$

proving the Lemma. □

The combination of the next Lemma and Lemma 4.6 entails that the Hurewicz homomorphism respects the product structures in an appropriate sense.

**Lemma 4.7.** *Suppose that the elements $x \in K_n^{\mathrm{rel}}(A)$ and $y \in K_m^{\mathrm{rel}}(B)$ are represented by the continuous maps*

$$f : S^n \to |R_p(A)|^+ \subseteq |R(A)|^+ \quad \textit{and} \quad g : S^m \to |R_q(B)|^+ \subseteq |R(B)|^+$$

*respectively. We then have the equality*

$$h_{n+m}(x *^{\mathrm{rel}} y) = (\gamma_{p,q}^{\mathrm{rel}} \circ (f \times g))_*(\zeta)$$

*in $H_{n+m}(|R(A \hat{\otimes} B)|^+)$. Here $\gamma_{p,q}^{\mathrm{rel}} : |R_p(A)|^+ \times |R_q(B)|^+ \to |R(A \hat{\otimes} B)|^+$ denotes the product map constructed in Section 3.3.*

*Proof.* By definition, the product $x *^{\mathrm{rel}} y \in K_{n+m}^{\mathrm{rel}}(A \hat{\otimes} B)$ is represented by the map

$$\widehat{\gamma}_{p,q}^{\mathrm{rel}} \circ (f \wedge g) : S^n \wedge S^m \to |R(A \hat{\otimes} B)|^+$$

Using (7) we thus get that the Hurewicz homomorphism of the product is given by

$$\begin{aligned} h_{n+m}(x *^{\mathrm{rel}} y) &= (\widehat{\gamma}_{p,q}^{\mathrm{rel}} \circ (f \wedge g))_*(\mathbf{1}_{n+m}) \\ &= (\widehat{\gamma}_{p,q}^{\mathrm{rel}} \circ (f \wedge g) \circ \pi)_*(\zeta) \\ &= (\widehat{\gamma}_{p,q}^{\mathrm{rel}} \circ \pi \circ (f \times g))_*(\zeta) \end{aligned}$$

The result of the lemma now follows by noting that the maps

$$\widehat{\gamma}_{p,q}^{\mathrm{rel}} \circ \pi \quad \text{and} \quad \gamma_{p,q}^{\mathrm{rel}} : |R_p(A)|^+ \times |R_q(B)|^+ \to |R(A \hat{\otimes} B)|^+$$

are homotopic. See Section 3.3. □

4.5. **The relative Chern character respects the exterior products.** We are now ready to prove the main result of this part of the paper: The counterpart in continuous cyclic homology of the exterior product in relative $K$-theory is given by the exterior product of degree one. The relevant multiplicative structures are described in Section 3.3 and Section 2.3.

Let $+ : HC_*(A) \oplus HC_*(A) \to HC_*(A)$ denote the addition on the continuous cyclic homology groups. Let $\pi : H_*(R(A) \times R(A)) \to H_*(R(A)) \oplus H_*(R(A))$ denote the map induced by the projection onto each factor. Furthermore, let $\oplus : R(A) \times R(A) \to R(A)$ denote the pointwise direct sum as introduced in Section 3.2. We will need the following preliminary result on the additive structures.

**Lemma 4.8.** *We have the equality*

$$+ \circ \big((\mathrm{TR} \circ \varepsilon \circ L) \oplus (\mathrm{TR} \circ \varepsilon \circ L)\big) \circ \pi = \mathrm{TR} \circ \varepsilon \circ L \circ \oplus_*$$

*between maps $H_n(R(A) \times R(A)) \to HC_{n-1}(A)$. Here we have suppressed the identification $H_n(R(A)) \cong H_n(R^\infty(A))$ of Lemma 3.2.*

*Proof.* The result is a consequence of the naturality of the involved maps and the behaviour of the generalized trace with respect to the direct sum operation. □

Using the work accomplished in Section 4.1, 4.2, 4.3 and 4.4 we are now able to prove the first main theorem of this paper.

**Theorem 4.9.** *For each $x \in K_n^{\mathrm{rel}}(A)$ and each $y \in K_m^{\mathrm{rel}}(B)$ we have the equality*

$$\mathrm{ch}^{\mathrm{rel}}(x *^{\mathrm{rel}} y) = \mathrm{ch}^{\mathrm{rel}}(x) * \mathrm{ch}^{\mathrm{rel}}(y)$$

*in $HC_{n+m-1}(A \hat{\otimes} B)$.*

*Proof.* Suppose that $x \in K_n^{\mathrm{rel}}(A)$ and $y \in K_m^{\mathrm{rel}}(B)$ are represented by the maps $f : S^n \to |R_p(A)|^+ \subseteq |R(A)|^+$ and $g : S^m \to |R_q(B)|^+ \subseteq |R(B)|^+$ respectively. By Lemma 4.7 we have

$$\begin{aligned} \mathrm{ch}^{\mathrm{rel}}(x *^{\mathrm{rel}} y) &= (\mathrm{TR} \circ \varepsilon \circ L \circ h_{n+m})(x *^{\mathrm{rel}} y) \\ &= (\mathrm{TR} \circ \varepsilon \circ L) \circ (\gamma_{p,q}^{\mathrm{rel}} \circ (f \times g))_*(\zeta) \end{aligned}$$

However it follows by definition of $\gamma_{p,q}^{\mathrm{rel}} : |R_p(A)|^+ \times |R_q(B)|^+ \to |R(A \hat{\otimes} B)|^+$ and by Lemma 4.6 and Lemma 4.8 that

$$\begin{aligned} &(\mathrm{TR} \circ \varepsilon \circ L) \circ (\gamma_{p,q}^{\mathrm{rel}} \circ (f \times g))_*(\zeta) \\ &\quad = (\mathrm{TR} \circ \varepsilon \circ L)(h_n(f) \times h_m(g)) + (\mathrm{TR} \circ \varepsilon \circ L)\big((V_* \circ \iota_*)(h_n(1_p) \times h_m(g))\big) \\ &\qquad + (\mathrm{TR} \circ \varepsilon \circ L)\big((V_* \circ \iota_*)(h_n(f) \times h_m(1_q))\big) + (\mathrm{TR} \circ \varepsilon \circ L)(h_n(1_p) \times h_m(1_q)) \end{aligned}$$

Here $V : |R(A \hat{\otimes} B)|^+ \to |R(A \hat{\otimes} B)|^+$ denotes the homotopy inverse of the $H$-group structure on $|R(A \hat{\otimes} B)|^+$, see Section 3.2. But the elements $h_n(1_p) \in H_n(R_p(A))$ and $h_m(1_q) \in H_m(R_q(B))$ are both trivial so we must have

$$\mathrm{ch}^{\mathrm{rel}}(x *^{\mathrm{rel}} y) = (\mathrm{TR} \circ \varepsilon \circ L)(h_n(f) \times h_m(g))$$

The conclusion of the Theorem now follows from a combination of the results in Theorem 4.3, Theorem 4.4 and Theorem 4.5. □

## 5. A calculation of the multiplicative character

We start this section by briefly recalling the construction of the multiplicative character as given in [12]. See also [16, 24].

Let $(F, H)$ denote an odd $2p$-summable Fredholm module over a unital Banach algebra $A$. To ease the exposition we will assume that the representation $\pi : A \to \mathcal{L}(H)$ and the map $a \mapsto [F, \pi(a)] \in \mathcal{L}^{2p}(H)$ are both continuous. We will always suppress the representation. Remark that the conditions on continuity are not necessary for the construction of the multiplicative character to work. They are however convenient for our exposition. See [12].

The continuous linear map

$$\tau_{2p-1} : C^\lambda_{2p-1}(A) \to \mathbb{C} \qquad (a_0, \ldots, a_{2p-1}) \mapsto \tfrac{c_p}{2^{2p}} \mathrm{Tr}(F[F, a_0] \cdot \ldots \cdot [F, a_{2p-1}])$$

where $c_p = (-1)^{p-1} \frac{(2p-1)!}{(p-1)!}$ then determines a continuous cyclic cocycle and consequently a homomorphism

$$\tau_{2p-1} : HC_{2p-1}(A) \to \mathbb{C}$$

See [10, 11]. The composition of this index cocycle with the relative Chern character thus yields a homomorphism

$$\tau_{2p-1} \circ \mathrm{ch}^{\mathrm{rel}}_{2p} : K^{\mathrm{rel}}_{2p}(A) \to \mathbb{C}$$

This is the additive character of the Fredholm module.

The next step in the construction then consists of showing that the image of the composition

$$\tau_{2p-1} \circ \mathrm{ch}^{\mathrm{rel}}_{2p} \circ v : K^{\mathrm{top}}_{2p+1}(A) \to \mathbb{C}$$

is contained in the additive subgroup $(2\pi i)^p \mathbb{Z} \subseteq \mathbb{C}$. Here the map $v : K^{\mathrm{top}}_{2p+1}(A) \to K^{\mathrm{rel}}_{2p}(A)$ is the boundary map of the long exact sequence (4) in Section 3. This is accomplished in [12, Section 4.10]. By consequence the additive character descends to a homomorphism

$$M_F : \mathrm{Coker}(v) \cong \mathrm{Im}(\theta) \to \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

This is the odd multiplicative character associated with the odd $2p$-summable Fredholm module $(F, H)$. With some further effort the multiplicative character can be extended to a map on algebraic $K$-theory, however we will only need the restriction to the subgroup $\mathrm{Im}(\theta) \subseteq K_{2p}(A)$ for our calculations. Again, remark that the multiplicative character defined in this section differs from the one given in [12] by the constant $(2p-1)!$ on $K_{2p}(A)$.

5.1. **The relative Chern character of a product of contractions.** Let $A$ be a *commutative*, unital Banach algebra. In this section we will give a concrete formula for the application of the relative Chern character to products of certain elements in relative $K$-theory. We will make use of the multiplicative properties of the relative Chern character which we investigated in Section 4.

We let

$$*^{\mathrm{rel}} : K^{\mathrm{rel}}_n(A) \times K^{\mathrm{rel}}_m(A) \to K^{\mathrm{rel}}_{n+m}(A) \quad \text{and} \quad * : HC_{n-1}(A) \otimes_{\mathbb{C}} HC_{m-1}(A) \to HC_{n+m-1}(A)$$

denote the (interior) products in relative $K$-theory and continuous cyclic homology. Note that these products are only available by the commutativity assumption on $A$. See Section 3.3 and Section 2.3.

Furthermore, for each $a \in M_\infty(A)$ we let $\gamma_a \in R(A)_1$ denote the smooth path defined by

$$\gamma_a(t) = e^{-ta} \quad \text{for all} \quad t \in [0,1]$$

**Theorem 5.1.** *Let $a_0, \ldots, a_{2p-1} \in M_\infty(A)$. The relative Chern character of the product*

$$x = [\gamma_{a_0}] *^{\mathrm{rel}} \ldots *^{\mathrm{rel}} [\gamma_{a_{2p-1}}] \in K_{2p}^{\mathrm{rel}}(A)$$

*is given by*

$$\mathrm{ch}^{\mathrm{rel}}(x) = \sum_{\mu \in \Sigma_{2p-1}} \mathrm{sgn}(\mu) \mathrm{TR}(a_0) \otimes \mathrm{TR}(a_{\mu(1)}) \otimes \ldots \otimes \mathrm{TR}(a_{\mu(2p-1)}) \in HC_{2p-1}(A)$$

*Proof.* By Theorem 4.9 we have

$$\mathrm{ch}^{\mathrm{rel}}(x) = \mathrm{ch}^{\mathrm{rel}}[\gamma_{a_0}] * \ldots * \mathrm{ch}^{\mathrm{rel}}[\gamma_{a_{2p-1}}] \in HC_{2p-1}(A)$$

Furthermore, the relative Chern character of the individual terms is given by

$$\mathrm{ch}^{\mathrm{rel}}([\gamma_a]) = (\mathrm{TR} \circ \varepsilon \circ L)([\gamma_a]) = \mathrm{TR}\Big(\int_0^1 \frac{d\gamma_a}{dt} \cdot \gamma_a^{-1} dt\Big) = -\mathrm{TR}(a)$$

for each $a \in M_\infty(A)$. The desired result now follows by definition of the product of degree one in continuous cyclic homology. □

5.2. **Products of commutators and the cyclic cocycle of A. Connes.** Our purpose is now to obtain a different expression for the evaluation of the index cocycle to certain elements in the cyclic homology group. We refer to the beginning of Section 5 for the definition of the index cocycle which was (of course) originally introduced by A. Connes, [10, 11].

Let $(F, H)$ be an odd $2p$-summable Fredholm module over a commutative, unital Banach algebra $A$. We will suppose that the representation $\pi : A \to \mathcal{L}(H)$ and the linear map $a \mapsto [F, \pi(a)] \in \mathcal{L}^{2p}(H)$ are continuous. Let $P = \frac{F+1}{2}$ denote the associated projection.

For each $n \in \mathbb{N}$, let $SE_n \subseteq \Sigma_n$ denote the subset of permutations defined by

$$s \in SE_n \Leftrightarrow \big(s \in \Sigma_n \text{ and } s(2i) < s(2i+1)\big)$$

We then have the following combinatorial result, which can be proved by induction.

**Lemma 5.2.** *For each algebra $B$ over $\mathbb{C}$ we have the identity*

$$\sum_{\mu \in \Sigma_{2p}} \mathrm{sgn}(\mu) x_{\mu(0)} \otimes \ldots \otimes x_{\mu(2p-1)} = \sum_{s \in SE_{2p}} \mathrm{sgn}(s) X_{s(0),\, s(1)} \otimes \ldots \otimes X_{s(2p-2),\, s(2p-1)}$$

*in the tensor product $B^{\otimes 2p}$. Here $X_{s(2i),\, s(2i+1)} \in B \otimes B$ denotes the commutator*

$$X_{s(2i),\, s(2i+1)} = x_{s(2i)} \otimes x_{s(2i+1)} - x_{s(2i+1)} \otimes x_{s(2i)}$$

Now, let $T \in C^{2p-1}(A)$ denote the cochain given by

$$T : (x_0, \dots, x_{2p-1}) \mapsto c_p \mathrm{Tr}(Px_0(1-P)x_1P \cdot \ldots \cdot Px_{2p-2}(1-P)x_{2p-1}P)$$

Here $c_p = (-1)^{p-1}\frac{(2p-1)!}{(p-1)!}$. We then note that $T \circ t^2 = T$ and that $T \circ (1+t) = \tau_{2p-1}$, where $\tau_{2p-1} \in C^{2p-1}_\lambda(A)$ is the index cocycle associated with the odd $2p$-summable Fredholm module $(F, H)$ over $A$. We can thus conclude that $T \circ N = p \cdot \tau_{2p-1}$.

**Theorem 5.3.** *For each $x_0, \dots, x_{2p-1} \in A$ we have the identity*

$$\begin{aligned}
&\sum_{\mu \in \Sigma_{2p-1}} \mathrm{sgn}(\mu)\tau_{2p-1}(x_0 \otimes x_{\mu(1)} \otimes \ldots \otimes x_{\mu(2p-1)}) \\
&\quad = (-1)^p c_p \sum_{s \in SE_{2p-1}} \mathrm{sgn}(s)\mathrm{Tr}([Px_0P, Px_{s(1)}P] \cdot \ldots \cdot [Px_{s(2p-2)}P, Px_{s(2p-1)}P])
\end{aligned}$$

*Proof.* Using Lemma 5.2 and the considerations preceeding the statement of the Theorem we get that

$$\begin{aligned}
&\sum_{\mu \in \Sigma_{2p-1}} \mathrm{sgn}(\mu)\tau_{2p-1}(x_0 \otimes x_{\mu(1)} \otimes \ldots \otimes x_{\mu(2p-1)}) \\
&\quad = \frac{1}{p} \sum_{\mu \in \Sigma_{2p}} \mathrm{sgn}(\mu) T(x_{\mu(0)} \otimes x_{\mu(1)} \otimes \ldots \otimes x_{\mu(2p-1)}) \\
&\quad = \frac{1}{p} \sum_{s \in SE_{2p}} \mathrm{sgn}(s) T(X_{s(0),\, s(1)} \otimes \ldots \otimes X_{s(2p-2),\, s(2p-1)})
\end{aligned}$$

However, by commutativity of $A$ we get the relation

$$[PxP, PyP] = -Px(1-P)yP + Py(1-P)xP \qquad \forall x, y \in A$$

It follows that

$$\begin{aligned}
&\sum_{\mu \in \Sigma_{2p-1}} \mathrm{sgn}(\mu)\tau_{2p-1}(x_0 \otimes x_{\mu(1)} \otimes \ldots \otimes x_{\mu(2p-1)}) \\
&\quad = \frac{(-1)^p}{p} c_p \sum_{s \in SE_{2p}} \mathrm{sgn}(s) \\
&\qquad \mathrm{Tr}([Px_{s(0)}P, Px_{s(1)}P] \cdot \ldots \cdot [Px_{s(2p-2)}P, Px_{s(2p-1)}P]) \\
&\quad = (-1)^p c_p \sum_{s \in SE_{2p-1}} \mathrm{sgn}(s) \\
&\qquad \mathrm{Tr}([Px_0P, Px_{s(1)}P][Px_{s(2)}P, Px_{s(3)}P] \cdot \ldots \cdot [Px_{s(2p-2)}P, Px_{s(2p-1)}P])
\end{aligned}$$

In the last equation we have used that the set of permutations $SE_{2p-1}$ can be viewed as the quotient of $SE_{2p}$ by an action of the cyclic group on $p$ elements. We have thus obtained the desired result. $\square$

5.3. **An evaluation of the multiplicative character on higher Loday symbols.** We are now ready to prove our concrete formula for the application of the multiplicative character to higher Loday products. This will accomplish the main purpose of the paper.

Let $(F, H)$ be an odd $2p$-summable Fredholm module over a commutative, unital Banach algebra $A$. We will suppose that the representation $\pi : A \to \mathcal{L}(H)$ and the linear map $a \mapsto [F, \pi(a)] \in \mathcal{L}^{2p}(H)$ are continuous. Let $P = \frac{F+1}{2}$ denote the associated projection. We refer to the beginning of Section 5 for a brief reminder on the construction of the multiplicative character.

**Theorem 5.4.** *Let $a_0, \ldots, a_{2p-1} \in M_\infty(A)$. The multiplicative character of the Loday product $[e^{a_0}] * \ldots * [e^{a_{2p-1}}] \in K_{2p}(A)$ is then given by*

$$\begin{aligned}
&M_F([e^{a_0}] * \ldots * [e^{a_{2p-1}}]) \\
&\quad = (-1)^p c_p \sum_{s \in SE_{2p-1}} \mathrm{sgn}(s) \\
&\qquad \mathrm{Tr}([P\mathrm{TR}(a_0)P, P\mathrm{TR}(a_{s(1)})P] \cdot \ldots \cdot [P\mathrm{TR}(a_{s(2p-2)})P, P\mathrm{TR}(a_{s(2p-1)})P]) \\
&\qquad \in \mathbb{C}/(2\pi i)^p\mathbb{Z}
\end{aligned}$$

*Proof.* For each $i \in \{0, \ldots, 2p-1\}$ we let $\gamma_{a_i} \in R(A)_1$ denote the smooth path given by $\gamma_{a_i} : t \mapsto e^{-ta_i}$. We then have

$$\theta([\gamma_{a_i}]) = [\gamma_{a_i}(1)^{-1}] = [e^{a_i}]$$

By Theorem 3.17 the map $\theta : \oplus_{n=1}^\infty K_n^{\mathrm{rel}}(A) \to \oplus_{n=1}^\infty K_n(A)$ is a homomorphism of graded rings, so we get that

$$\theta([\gamma_{a_0}] *^{\mathrm{rel}} \ldots *^{\mathrm{rel}} [\gamma_{a_{2p-1}}]) = [e^{a_0}] * \ldots * [e^{a_{2p-1}}]$$

By definition of the multiplicative character we then have

$$M_F([e^{a_0}] * \ldots * [e^{a_{2p-1}}]) = (\tau_{2p-1} \circ \mathrm{ch}^{\mathrm{rel}})([\gamma_{a_0}] *^{\mathrm{rel}} \ldots *^{\mathrm{rel}} [\gamma_{a_{2p-1}}]) \in \mathbb{C}/(2\pi i)^p\mathbb{Z}$$

But it follows from Theorem 5.1 and Theorem 5.3 that the right hand side is given by

$$\begin{aligned}
&(\tau_{2p-1} \circ \mathrm{ch}^{\mathrm{rel}})([\gamma_{a_0}] *^{\mathrm{rel}} \ldots *^{\mathrm{rel}} [\gamma_{a_{2p-1}}]) \\
&\quad = \sum_{\mu \in \Sigma_{2p-1}} \mathrm{sgn}(\mu) \tau_{2p-1}(\mathrm{TR}(a_0) \otimes \mathrm{TR}(a_{\mu(1)}) \otimes \ldots \otimes \mathrm{TR}(a_{\mu(2p-1)})) \\
&\quad = (-1)^p c_p \sum_{s \in SE_{2p-1}} \mathrm{sgn}(s) \\
&\qquad \mathrm{Tr}([P\mathrm{TR}(a_0)P, P\mathrm{TR}(a_{s(1)})P] \cdot \ldots \cdot [P\mathrm{TR}(a_{s(2p-2)})P, P\mathrm{TR}(a_{s(2p-1)})P])
\end{aligned}$$

proving the desired result. □

**Corollary 5.5.** *For any commutative unital Banach algebra the multiplicative character is calculizable on the subgroup of $K_{2p}(A)$ generated by Loday products of elements in the connected component of the identity, $GL_0(A)$.*

*Proof.* Since each element in $GL_0(A)$ can be obtained as a product of exponentials, the result follows by noting that the Loday product is multilinear and that the multiplicative character is a homomorphism of abelian groups. □

## References


[1] C. A. Berger, B. I. Shaw, *Selfcommutators of multicyclic hyponormal operators are trace class*, Bull. Amer. Math. Soc. (N.S.) **79**, 1973, , 1193-1199.

[2] C. A. Berger, B. I. Shaw, *Intertwining, analytic structure, and trace norm estimate*, Proc. Conf. Operator Theory, Lecture Notes in Math., **345**, Springer-Verlag, Berlin, 1973, 1-6.

[3] A. J. Berrick, *Caracterization of Plus-Constructive Fibrations*, Advances in Mathematics **48**, 1983, 172-176.

[4] L. G. Brown, *The Determinant Invariant for Operators with Trace Class Self Commutators*, Proc. Conf. Operator Theory, Lecture Notes in Math., **345**, Springer-Verlag, Berlin, 1973, 210-228.

[5] L. G. Brown, *Operator Algebras and Algebraic $K$-theory*, Bull. Amer. Math. Soc., **81**, 1973, 1119-1121.

[6] R. W. Carey, J. D. Pincus, *Steinberg Symbols Modulo the Trace Class, Holonomy, and Limit Theorems for Toeplitz Determinants*, Transactions of the American Mathematical Society, vol. **358**, 2004, 509-551.

[7] R. W. Carey, J. D. Pincus, *Perturbation Vectors*, Integral Equations Operator Theory, **35**, , 1999, 271-365.

[8] R. W. Carey, J. D. Pincus, *Commutators, symbols and determining functions*, J. Funct. Analysis, **19**, 1975, 50-80.

[9] R. W. Carey, J. D. Pincus, *Mosaics, principal functions, and mean motion in von Neumann algebras*, Acta. Math. **138**, 1977, 153-218.

[10] A. Connes, *Noncommutative differential geometry* Part I: The Chern character in $K$-homology, Preprint I.H.E.S. (M/82/53), 1982. Part II: de Rham homology and noncommutative algebra, Preprint I.H.E.S. (M/83/19), 1983.

[11] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[12] A. Connes and M. Karoubi, *Caractère multiplicatif d'un module de Fredholm*, K-theory **2**, 1988, 431-463.

[13] A. Grothendieck, *Produits Tensoriels Topologiques*, Mem. AMS., **16**, 1955.

[14] J. W. Helton and R. E. Howe, *Integral Operators: Commutators, Traces, Index, and Homology*, Proc. Conf. on Operator Theory, Lecture Notes in Math., **345**, Springer-Verlag, Berlin, Heidelberg, New York, 1973, 141-209.

[15] J. Kaad, *Comparison of secondary invariants of finitely summable Fredholm modules*, submitted for publication.

[16] J. Kaminker, *Algebraic $K$-theory invariants for operator theory*, Contemporary Mathematics, **185**, Amer. Math. Soc., Providence, RI, 1995, 187-194.

[17] M. Karoubi, *Homologie Cyclique et $K$-théorie*, Astérisque **149**, 1987.

[18] J. -L. Loday, *$K$-théorie algébrique et représentations de groupes*, Ann. Scient. Ecole Norm. Sup. **9**, 1976, 309-377.

[19] J. -L. Loday, *Cyclic Homology*, Grundlehren der mathematischen Wissenschaften, **301**, Springer-Verlag, New York, Berlin, Heidelberg, 1992.

[20] J. -L. Loday and D. Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment Math. Helv. **59**, 1984, 565-591.

[21] J. P. May, *Simplicial Objects in Algebraic Topology*, Chicago Lectures in Mathematics, 1993.

[22] D. Quillen, *Higher Algebraic $K$-theory*, Lecture Notes in Maths **341**, 1973, 85-147.

[23] J. Rosenberg, *Algebraic $K$-theory and Its Applications*, Graduate Texts in Math., **147**, Springer-Verlag, New York, Berlin, Heidelberg, 1994.

[24] J. Rosenberg, *Comparison Between Algebraic and Topological $K$-Theory for Banach Algebras and $C^*$-Algebras*, Handbook of Algebraic $K$-theory, Springer, 2004, 843-874.

[25] J. Stasheff, *$H$-spaces from a Homotopy Point of View*, Lecure notes in Mathematics, **161**, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[26] U. Tillmann, *Factorization of the (Relative) Chern Character through Lie Algebra Homology*, K-theory **6**, 1992, 457-463.

[27] B. Tsygan, *Homology of Lie algebras of matrices over rings and Hochschild homologies*, Uspechi Mat. Nauk. **38**, 1983, 217-218.

[28] B. Tsygan, *Cyclic homology*, unpublished.

[29] J. B. Wagoner, *Delooping Classifying Spaces in Algebraic $K$-theory*, Topology, **11**, 1972, 349-370.